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# Random partnerships in spatial game theory

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We study the effects of random partnerships, introduced to interaction or replacement networks, on the evolution of cooperation in spatial game theory on a square lattice. For the spatial prisoner dilemma game, we show that, compared with the case without random partnership, cooperation can be enhanced regardless of whether a random partnership is introduced to an interaction or replacement network. Specifically, the enhancement of cooperation is strongest in the limit of zero randomness. We show explicitly that the cooperator frequency is a decreasing function of randomness, and the cooperation eventually vanishes once the randomness is strong enough. For the spatial snow drift game, we find that the enhancement of cooperation occurs only when a random partnership is introduced to an interaction network.

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## I. INTRODUCTION

Natural selection in Darwinian theory favors egoists who try to maximize their individual benefits even at the expense of the group. However, the evidence in biology and sociology shows that cooperation among selfish individuals may arise and be sustainable [1]. Resolving the social dilemma in which a conflict arises between the benefits of individuals the group is a challenging problem not only for biologists and sociologists but also for physicists.

Since its introduction, the evolutionary prisoner's dilemma (PD) [2–4] has been the basis of understanding cooperative behavior in populations of selfish individuals. In the original one-shot two-player game, players can follow one of two options: defection (D) and cooperation (C). Each player's payoff depends on the choices of both players. For example, two cooperators receive the reward $R$ and two defectors receive the punishment $P$, while the cooperator receives the sucker's payoff $S$ at the same time his opponent exploits the temptation $T$ by choosing defection. The ranking of $T > R > P > S$ is required for the PD game. For the rational players, mutual defection is the Nash equilibrium in the PD game. In the iterated PD game, the assumption that the mutual cooperation gains the highest total payoff imposes another constraint $2R > T + S$. When the population is infinite and well-mixed, the mean-field description of the evolutionary PD game in which the density of cooperators follows $\dot{\rho}_c = \rho_c(A_C - \bar{A})$, where $A_C$ refers to the payoff of a cooperator and $\bar{A}$ the average population payoff, leads to the extinction of cooperators.

In the last few years, several mechanisms favoring cooperation among selfish individuals have been proposed, such as kin selection [5], reciprocity [6], reputation [7], voluntary participation [8], and structured populations [9,10]. Among these, the structured populations have drawn a lot of attention. In the spatial evolutionary PD game introduced by Nowak and May [9], individuals located on a lattice play with their neighbors. At each round, players interact with their neighbors by choosing cooperation or defection, and the sum of the encounters from each neighbor gives the payoff for a certain player. In the next move, each player follows the most successful neighbor. Given this spatial evolutionary game, Nowak and May have shown that cooperation can emerge through the way that cooperators form clusters to resist exploitation by defectors. The spatial evolutionary PD game in Ref. [9] is a deterministic cellular automata and can be transformed to a stochastic model in different means. Szabo and his colleagues introduced noise to the strategy adoption process [11,12]. They studied the effect of dynamical randomness on the stationary concentration of cooperators, and found the phase transition involving the extinction of cooperators or defectors to be in the directed percolation university class. Perc and his collaborators introduced noise to the payoff matrix [13–15]. The reported results indicate a stochastic resonance where the frequency of cooperators reaches its maximum at an intermediate noise strength. The noise can also be introduced to the structure of the underlying lattice. Along these lines, the evolutionary PD game on the small-world, regular random, scale-free networks, hierarchical lattice, and on the co-evolving networks has been explored extensively [16–24]. In particular, Santos and Pacheco pointed out that the scale-free network can provide a unifying framework for the enhancement of cooperation [16].

In this work, we consider another method to bring randomness into the deterministic spatial game. The system under investigation is the spatial PD game on a two-dimensional (2D) lattice with periodic boundary conditions. At each time step, players have a chance to randomly choose partners. By using Monte Carlo simulations, we investigate the influences of random partnership on cooperation and find that randomness favors cooperation in a large range of parameters. Specifically, cooperation is greatly enhanced for weak randomness. In the end, we also briefly report the effects of random partnership on the spatial snow drift (SD)

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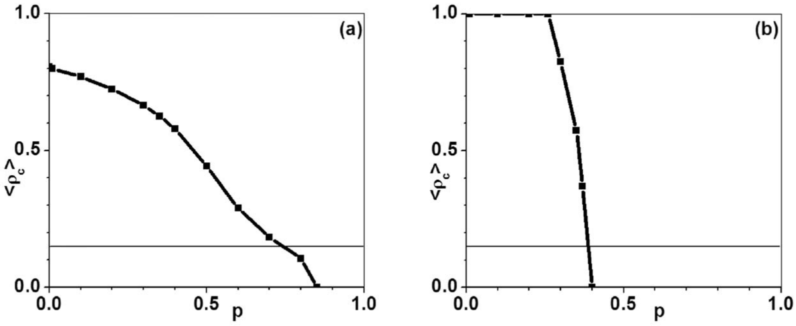

FIG. 1. The equilibrium value of the cooperator frequency, $\langle\rho_c\rangle$, is plotted against $p$ for the case of RPR in (a) and for the case of RPI in (b). The equilibrium cooperator frequency, $\langle\rho_{c,d}\rangle = 0.15$, in the absence of random partnership is presented by a solid line in these plots. It is clear from the two plots that $\langle\rho_c\rangle$ is discontinuous at $p=0$.

game on a 2D lattice where the SD game is constrained by the ranking of $T>R>S>P$.

## II. MODEL

As pointed out in Refs. [25,26], there are two types of networks determining the population structure in the spatial PD game: the interaction network, in which each edge indicates that the players on its ends can play the PD game with each other, and the replacement network, which defines who-is-the-role-model-of-whom in the evolutionary updating. In this work, we consider two situations: the random partnership is introduced to the interaction network (RPI), or the random partnership is introduced to the replacement network (RPR).

We consider a square lattice with periodic boundary conditions where each player occupies a site. Without random partnership, players perform the PD game with their nearest neighbors on the lattice in each round. The total payoff of each player in each round is the sum of gains from all interactions in which he participates. Then each player adopts the strategy of the neighbor with the highest payoff. In the case of RPR, each player performs the PD game with his nearest neighbors on the 2D lattice as normal. However, in the stage of updating strategy, the player has a probability of $p$ to randomly choose one partner from the whole population and adopts his previous strategy if he has a higher payoff. In the case of RPI, the random partnership enters the interaction network. That is, each player has a probability of $p$ to play the game with four neighbors randomly chosen from the whole population. Then in the stage of updating strategy, each player behaves as that on the 2D lattice.

Throughout this work, we set $T=1.2$, $R=1$, $P=0$, and $S=-0.5$ unless specified. The dimension of the lattice is set to be $L=100$.

## III. RESULTS

We first consider the case of RPR. Simulations start with an equal percentage of cooperators and defectors that are randomly distributed among the population. The important quantity to characterize cooperation is the cooperator frequency $\rho_c$. In the absence of random partnership (i.e., in a deterministic spatial PD game), the cooperator frequency reaches an equilibrium value $\rho_{c,d}=0.15$ after a transient. Now we introduce random partnership to the replacement network.

Generally, for any given initial condition, the system will reach a statistical equilibrium, where the instantaneous cooperator frequency $\rho_c$ fluctuates around an equilibrium value, after a transient that is much longer for extremely low $p$. We calculate the equilibrium value $\langle\rho_c\rangle$ by averaging $\rho_c$ over a long time interval, for example $\Delta t=10^5$, after the transient. The dependence of $\langle\rho_c\rangle$ with randomness $p$ is plotted in Fig. 1(a), which shows that $\langle\rho_c\rangle$ monotonically decreases with $p$. It is interesting to find that, comparing with the deterministic case ($p=0$), the cooperation could be enhanced by the introduction of a random partnership (that is, $\langle\rho_c\rangle$ is larger than $\rho_{c,d}$). What is more, the enhancement of cooperation is sustained in a large range of $p$, for example $0<p<0.73$, and the mean-field approximation of the evolutionary PD game is recovered only when $p$ is larger than a critical value ($p_c \simeq 0.85$). It should be noted that the maximum of $\langle\rho_c\rangle$ is obtained in the limit of $p\to 0$, and the dependence of $\langle\rho_c\rangle$ on $p$ is not continuous at $p=0$, which implies that the spatial evolutionary game in its deterministic version is unstable to the disturbance of random partnerships.

Now we consider the case of RPI. The equilibrium values of the cooperator frequency, $\langle\rho_c\rangle$, are plotted in Fig. 1(b). Similar to the case of RPR, the enhancement of cooperation induced by random partnerships is obvious, and the equilibrium value of $\langle\rho_c\rangle$ is a monotonic function of $p$ too. However, different from the case of RPR, there are two thresholds: $p_{c,1}\simeq 0.26$ and $p_{c,2}\simeq 0.4$. When $p<p_{c,1}$, all of the defectors can be eliminated completely, that is, $\langle\rho_c\rangle=1$ in contrast with $\langle\rho_c\rangle\simeq 0.8$ in the limit of zero randomness in the case of RPR. The mean-field approximation is recovered above $p>p_{c,2}$ which is much smaller than that in the case of RPR. Between $p<p_{c,1}$ and $p<p_{c,2}$, $\langle\rho_c\rangle$ decreases sharply.

Before we move forward, some remarks related to the previous relevant works should be addressed. First, resonancelike behaviors in spatial game theory have been reported in Refs. [13,29]. In particular, in Ref. [29], the authors claimed that the spatial PD game on a Watts-Strogatz small-world network [30] can be considered as a noise-implemented game in its original regular lattice, where they found significant improvement of cooperation when an optimal amount of shortcut probability is assumed. Though at any given time the underlying networks in our work can also be regarded as a Watts-Strogatz small world network, our work is different from Ref. [29] for two reasons: (i) the structure of the underlying network in our work is not frozen, instead it varies with time; (ii) though we are investigating the enhancement of cooperation due to random partnerships, there are no resonancelike behaviors to find. The highest extent of coopera-

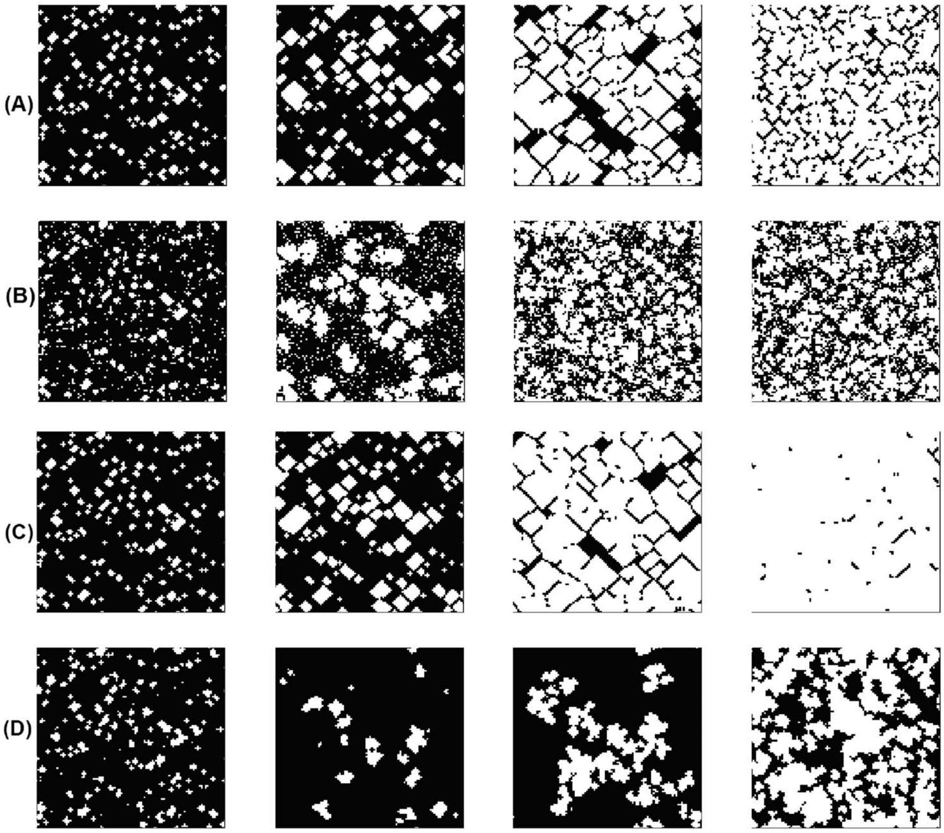

FIG. 2. The snapshots of equilibrium configurations of cooperators (white) and defectors (black) at different times. The time arrow goes from left to right. (a) $p=10^{-4}$ in the case of RPR; (b) $p=0.5$ in the case of RPR; (c) $p=10^{-4}$ in the case of RPI; (d) $p=0.36$ in the case of RPI.

tion is reached in the limit of zero randomness, and the increase of randomness always weakens the cooperation gradually. Second, in Refs. [25,26], the authors pointed out that it is always harder for cooperators to evolve whenever the interaction network and the replacement network do not coincide. However, there are some recent works claiming that the strongest cooperation could happen when the two networks are not the same [27,28]. In our work, the results presented in Fig. 1 show that the coincidence between the interaction and replacement networks is not necessary for enhancing the cooperation and that, more interestingly for the PD game, breaking the coincidence between these two networks may pose a discontinuous transition on the cooperation, which indicates that the strongest cooperation occurs in the limit of coincidence between the interaction and replacement networks.

Furthermore, we investigate how the spatial configurations of cooperators and defectors on the 2D lattice evolve. Starting with the deterministic version ($p=0$) and using its steady state as an initial condition, we plot the snapshots at successive times for different $p$ in Fig. 2. For weak randomness, we find that, in both cases of RPR and RPI, the enhancement of cooperation is ascribed to the growth of the clusters formed by the cooperators, and the growth of these cooperator clusters is in the direction perpendicular to the boundary of the clusters. The difference between the cases of RPR and RPI lies in the fact that, when statistical equilibrium is reached, the defectors in the case of RPR form short thin lines that always exist, though they vary with time. In contrast, the defectors in the case of RPI are eliminated in the end. For strong randomness ($p=0.5$ in the case of RPR and $p=0.37$ in the case of RPI), the enhancement of cooperation is also assisted by the growth of the cooperator clusters in both cases. However, different from the weak randomness situation, the evolution here is accompanied by the violent dynamics of the clusters, for example the generation or elimination of the cooperator clusters, and the drift of these clusters on the lattice.

These observations can be understood as follows. Basically, there are two time scales in the model we studied: $\tau_e$, which characterizes the evolution of the spatial game on the 2D lattice; and $\tau_r$, which characterizes how frequently the players are subjected to random partnership. $\tau_e$ is usually less than tens of time steps, while $\tau_r$ depends on the randomness parameter $p$. For extremely weak randomness ($p \ll 1$), $\tau_r$ is much larger than $\tau_e$ and the occurrences of random partnership become rare events that lead to the intermittency-like behavior in the evolution of the system. That is, the system first evolves fast to a steady state where the cooperators form clusters to resist the exploitation of the defectors; then a random partner is chosen by a player at a certain time step, and such an event is likely to start another relaxation process. It is noted that the occasional relaxation processes induced by random partnership play important roles in the enhancement of cooperation for weak randomness. To see it, we first consider the case of RPR for extremely weak randomness. Suppose that the system is in a steady state. Then once a random partnership is experienced by a player, the player is likely to replace its current strategy under any of three conditions: (i) the player is in the domain of the defectors; (ii) the player locates at the boundaries of the cooperator clusters; (iii) the player lies inside a cooperator cluster if there exist isolated

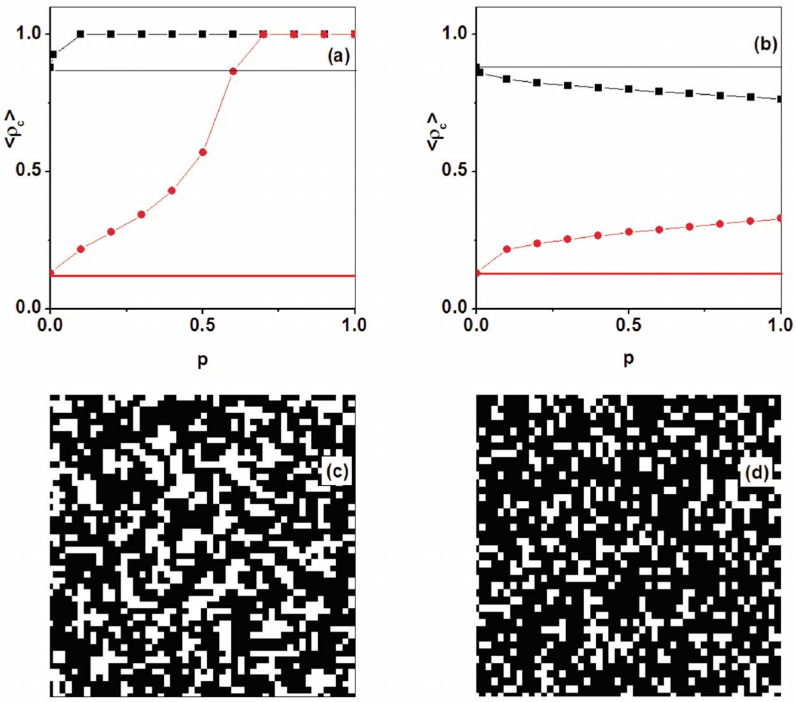

FIG. 3. (Color online) The equilibrium value of the cooperator frequency, $\langle\rho_c\rangle$, is plotted against $p$ for the case of RPI in (a) and for the case of RPR in (b). The black (or red) line with solid squares (or circles) is for the parameter sets $R=1$ and $r=0.2$ (or $R=1$ and $r=0.62$). The equilibrium cooperator frequency $\langle\rho_{c,d}\rangle=0.88$ for $R=1$ and $r=0.2$ are represented by a black line and $\langle\rho_{c,d}\rangle=0.13$ for $R=1$ and $r=0.62$ by a red line. For the parameter set of $R=1$ and $r=0.62$, the snapshots of equilibrium configurations of cooperators (white) and defectors (black) at steady state, at $p=0.3$, are presented for RPI in (c) and for RPR in (d). To be clear, only one-quarter of the space area is shown here.

defectors. Under the first condition, the player will shift his strategy back to defection once he follows his random neighbor inside a cooperator cluster, which means that such an event does not really contribute to the cooperator frequency. Under the second condition, both a cooperator becoming a defector when the teacher defector locates at the boundaries (C→D) and a defector becoming a cooperator when the teacher cooperator is inside the cluster (D→C) can happen. Considering the fact that the players inside the cooperator clusters are most likely to have a higher payoff (3R) than the defects unless the defects are isolated, the events of D→C are dominant over those of C→D. Consequently, the cooperator frequency increases. When the cooperator frequency becomes large enough, the domain of the defectors shrinks to be in the form of a large amount of isolated defectors [see the thin line shown in Fig. 2(a)], which will survive due to their highest payoff, $4T$. In the meantime, the events of C→D under the third condition occur. The interaction between the isolated defectors and the other cooperators has two effects: one is favorable for the survival of defectors when the players inside the cooperator clusters follow the isolated defectors due to the random partnership; the other is that the isolated defectors may turn their nearest neighbors to defectors, which will lower the defectors' payoffs and is unfavorable for the survival of defectors. The balance between these two effects gives the equilibrium value of $\rho_c$. Furthermore, it can be found that two factors are responsible for the decreasing tendency of $\rho_c$ with $p$: (i) increasing randomness will drive more players inside the cooperator clusters to change their strategies by following the isolated defects; (ii) increasing the randomness decreases the time scale $\tau_r$, which causes random partnerships experienced by different players to be possibly correlated. Such collectivity plays the role not only of having the influence of the events under the first condition on $\rho_c$ to be sustainable, but also making the events under the second condition more frequent by thickening the boundaries of the cooperator clusters. The thickened boundaries of the cooperator clusters tend to have the events of $C\rightarrow D$ to be dominant over those of $C\rightarrow D$, while the events of players changing strategies under the first condition are helpful for cooperation. It is the competition among the events of players changing strategies under the three conditions that leads to a gradually decreasing function of $\langle\rho_c\rangle$ against the randomness parameter $p$.

The above analysis can be applied to the case of RPI too. However, it is crucial to note that, in this case, the events of players changing their strategies happen only when these players locate at the boundaries of the cooperator clusters. It is also crucial to note that the isolated defectors cannot resist the invasion of the cooperators since the isolated defectors can only recruit their nearest neighbors, which is unfavorable for the survival of defectors. These two factors, which are different from the case of RPR, result in $\rho_c=1$ for weak randomness. For high randomness, the state of $\rho_c=0$ arises much earlier in the case of RPI than RPR due to the absence of the events of players changing their strategies under the first condition.

It should be noted that the greatly enhanced cooperation in this work can only be observed for $T<4R/3$, since when $T<4R/3$, only the isolated defectors can resist the invasion of the cooperators due to the random partnership. However, if $T$ is increased beyond $4/3R$, not only the isolated defectors but also the dimers consisting of two defectors could survive since they have higher payoffs than those of players inside the cooperator clusters, which leads to a sharp decrease of $\langle\rho_c\rangle$ even for weak randomness. Similarly, for larger $T$, there are more configurations for defectors to survive under the influences of random partnership, which further decrease cooperation among players.

Hauert and Doebeli pointed out [31] that the square lattice actually inhibits the evolution of cooperation in the spatial

SD game for a wide range of parameter since, in the spatial SD game, the cooperators form filamentlike clusters that generate an advantage for defectors. Now, we turn our attention to the spatial SD game on a square lattice with random partnership. The SD game can be reformulated as $T=b$, $R=b-c$, $S=b-c$, and $P=0$ with two positive values of $b$ and $c$. According to the replicator dynamics [4], the equilibrium frequency of cooperators in the well-mixed SD game is $1-r$, where $r=c/(2b-c)$ is the cost-to-benefit ratio of mutual cooperation. We consider two parameter sets: (i) $R=1$ and $r=0.2$, where $\langle\rho_c\rangle \simeq 0.88$ in the deterministic spatial SD game and $\langle\rho_c\rangle=0.8$ in the well-mixed SD game; (ii) $R=1$ and $r=0.62$, where $\langle\rho_c\rangle \simeq 0.13$ in the deterministic spatial SD game and $\langle\rho_c\rangle=0.38$ in the well-mixed SD game. The results are presented in Fig. 3. Different from the spatial PD game, the enhancement of cooperation in the spatial SD game can be found only when the random partnership is introduced to the interaction network. The reason can be seen in the spatial configurations of cooperators and defectors where the filamentlike clusters are reserved in the case of RPR and the compact clusters become prevalent in the case of RPI. From the relationship between $\langle\rho_c\rangle$ and $p$, we also find that the well-mixed limit cannot be reached by simply setting $p=1$ in the case of RPI or RPR as the spatial PD game does, which indicates that the random partnership in the spatial SD game could play more flexible roles in enhancing cooperation than in the spatial PD game.

## IV. CONCLUSIONS

In conclusion, we have investigated the spatial game theory on a 2D lattice with random partnership. We investigated two different cases: the random partnership introduced to the interaction network or to the replacement network. For the spatial PD game, we found that, in both cases, cooperation could be enhanced by a random partnership and the enhancement of cooperation is the strongest in the limit of zero randomness. Furthermore, we found that the enhancement of cooperation could be sustained in a large range of randomness, and cooperation is eliminated only for sufficiently large randomness. For the SD game, the enhancement of cooperation is found only when the random partnership is introduced to the interaction network.

## ACKNOWLEDGMENTS

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